On Maximum Principles for Monotone Matrices

Gisbert Stoyan
Computing Center of the Budapest University
Bogdánffy u.10/b
1117 Budapest, Hungary

Submitted by Miroslav Fiedler

ABSTRACT

For monotone matrices and H-matrices, maximum principles are defined, and necessary and sufficient conditions are derived for these principles to hold.

1. INTRODUCTION

Discrete maximum principles are of importance in the study of approximations to differential equations [4, 17]. For matrices of a certain structure corresponding to first-kind boundary-value problems, in [3] necessary and sufficient conditions have been derived which include that the matrix is monotone (of monotone kind) in the sense of [5], i.e. nonsingular with nonnegative inverse. Quite analogous properties of the solutions to linear equations the matrix of which is an irreducible M-matrix have been shown to hold in [19] in connection with the study of the open Leontief input-output model (and these results, in a special case, can be traced back through [1] and [12] back to [10]).

The discrete maximum principle as well as the above properties of open linear systems can be stated roughly as follows: The maximum response takes place in a part of the system where there is a nonzero influence. Hence, maximum principles should obviously be a property of most linear systems reasonably modeling physical, biological, economic, or technological processes.

In this paper, continuing work started in [20], we try to handle maximum principles as matrix properties which are independent of the interpretation of the original problem or of an explicit structure of the matrix such as a partitioning or irreducibility.

Firstly, a useful sufficient condition is established which is weaker than that obtained in [20], and necessary and sufficient conditions for the maxi-

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mum principle [20] are given in geometrical terms. Moreover, this principle is shown to be meaningful for certain singular matrices. We investigate also the relation to the maximum principle [3]. The concepts and results of the paper are illustrated by a number of examples. Finally, a maximum principle for H-matrices is obtained.

2. PRELIMINARIES

We shall use the following notation: $A=(a_{ij})$ is an $n\times n$ matrix; y, f, etc. are n-dimensional column vectors. Zero matrices, zero vectors, and the number zero are denoted by the same symbol 0; inequality relations for vectors or matrices are to be understood componentwise. Further,

$$N:=\{1,\dots,n\},\qquad N^{+}(f):=\{j\in N,\ f_j>0\},$$
$$N^{0}(f):=\{j\in N,\ f_j=0\},$$
$$A^{(-)}:=A-A^{(+)},\qquad A^{(+)}:=\begin{cases}a_{ij}, & i\neq j,\quad a_{ij}>0,\\ 0 & \text{else},\end{cases}$$

e^{i} is the ith coordinate unit vector, and $e^{(n)}$ the n-vector all components of which are equal to 1.

In [20], the matrix A was said to satisfy the maximum principle if

$$Ay=f \tag{1}$$

and $f\geqslant 0$ implies $y\geqslant 0$, and moreover

$$\max_{i\in N} y_i=\max_{j\in N^{+}(f)} y_j. \tag{2}$$

Here, the right-hand side is defined to be zero if $N^{+}(f)=\varnothing$.

The following result was obtained in [20]: If there exists an $\varepsilon_0>0$ such that $A+\varepsilon I$ is monotone for $\varepsilon\in[0,\varepsilon_0]$ and if $A^{(-)}e^{(n)}\geqslant 0$, then A satisfies the maximum principle.

3. A NEW SUFFICIENT CONDITION

We give now a more general sufficient condition for a matrix A to satisfy the maximum principle.

THEOREM 1. *Let A be monotone with $A^{(-)}$ either nonsingular, or singular and irreducible. Further, let $A^{(-)}e^{(n)} \geqslant 0$. Then A satisfies the maximum principle.*

Proof. Assume the contrary. Then there is an $f \geqslant 0, \neq 0$, such that

$$y_{\max} := \max_{i \in N} y_i > \max_{j \in N^+(f)} y_j.$$

Here y is the solution of (1), which exists uniquely and is nonnegative due to the monotonicity of A. Hence, $y_k = y_{\max}$ implies $k \in N^0(f)$, that is (summation being performed from 1 to n),

$$0 = f_k = \sum a_{kl} y_l = a_{kk} y_k + \sum_{\substack{l \neq k \\ a_{kl} < 0}} a_{kl} y_l + \sum_{\substack{l \neq k \\ a_{kl} > 0}} a_{kl} y_l$$

$$\geqslant y_k \Bigg(a_{kk} + \sum_{\substack{l \neq k \\ a_{kl} < 0}} a_{kl} \Bigg) + \sum_{\substack{l \neq k \\ a_{kl} > 0}} a_{kl} y_l \geqslant 0, \tag{3}$$

by $A^{(-)}e^{(n)} \geqslant 0$ and $y \geqslant 0$. Since equality must hold in (3), it follows for $M := \{ j \in N,\ y_j = y_{\max} \}$ that if $k \in M$, then $a_{kl} < 0$ implies $l \in M$,

$$a_{kk} + \sum_{\substack{l \neq k \\ a_{kl} < 0}} a_{kl} = 0, \tag{4}$$

and $a_{kl} > 0$ implies $y_l = 0$.

Let m be the number of indices in M, which is at least 1 and at most $n-1$ [else $f = 0$ because $M \subseteq N^0(f)$]. Observe that whether a matrix satisfies the maximum principle is not influenced by simultaneous rearrangements of equally indexed rows and columns. Hence, we may rearrange (1) so as to obtain

$$\begin{pmatrix} A_1 & B_2 \\ B_1 & A_2 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} f^1 \\ f^2 \end{pmatrix} \tag{5}$$

with m-vectors y^1 and $f^1 = 0$, and with square matrices A_1 and A_2. Here, $y_k^1 = y_{\max}$, $k = 1, \ldots, m$, and $y_k^2 < y_{\max}$, $k = m+1, \ldots, n$.

Suppose first $m = 1$. Then there are no negative entries in the first row of the matrix in (5), and further, the first diagonal entry is zero, by (4). But then

$A^{(-)}$ contains a zero row, which contradicts the assumptions. In general, for $m \geqslant 1$, we obtain via (4) that A_1 contains all negative entries of the first m rows. Thus, after rearranging, $A^{(-)}$ is seen to be of the form

$$\begin{pmatrix} A_1 & 0 \\ \cdot & \cdot \end{pmatrix}.$$

Here $A_1 e^{(m)} = 0$, due to (4), i.e., $A^{(-)}$ is singular and reducible, once again a contradiction, which proves the theorem. ■

REMARKS.

(1) Let there exist a majorizing element to $A^{(-)}$, that is, a vector $x > 0$ satisfying $A^{(-)}x > 0$. Since $A^{-1} \geqslant 0$ and $A \geqslant A^{(-)}$, it follows from [18] that $A^{(-)}$ is nonsingular, $A^{(-)} \geqslant 0$, and $A^{-1} \leqslant (A^{(-)})^{-1}$. Therefore, $A^{-1} \geqslant 0$ and the existence of a majorizing element of $A^{(-)}$ ensure nonsingularity of $A^{(-)}$ and an estimate of $y = A^{-1}f$ by $z = (A^{(-)})^{-1}f$. Moreover, writing $0 \leqslant A^{(-)}x = A^{(-)}De^{(n)}$, $D := \mathrm{diag}(x_i)$, from Theorem 1 it follows that AD satisfies the maximum principle. In case $x = e^{(n)}$, this implies the maximum principle to hold for A.

(2) Let A be a monotone matrix which is quasidominant in the sense of [13], i.e., $a_{ii} > 0$ holds for all i and $M(A) := A^{(-)} - A^{(+)}$ is an M-matrix. Then $A \geqslant A^{(-)} \geqslant M(A)$ and $0 \leqslant A^{-1} \leqslant (A^{(-)})^{-1} \leqslant M^{-1}(A)$, again by [18], taking for x the sum of the columns of $M^{-1}(A)$. Hence, a monotone quasidominant matrix A satisfies the maximum principle if $A^{(-)}e^{(n)} \geqslant 0$.

(3) The conditions of [20] for the maximum principle to hold (see Section 2) are stronger than the above ones: If $A^{-1} \geqslant 0$ and $A^{(-)}e^{(n)} \geqslant 0$, then $(A + \varepsilon I)^{-1} \geqslant 0$ for some nontrivial interval $[0, \varepsilon_0]$ may happen or not, independently of $A^{(-)}$ being nonsingular, or singular and irreducible, or singular and reducible (see the examples in Section 8 below).

4. NECESSARY AND SUFFICIENT CONDITIONS: GEOMETRICAL INTERPRETATION

Let A be a monotone matrix with $A^{-1} = (\alpha_{ij})$. We remark first that if A satisfies the maximum principle, then so does DA for any diagonal matrix D with positive diagonal entries. We put

$$D = \mathrm{diag}(d_1, \dots, d_n), \quad d_j := \sum_{i=1}^{n} \alpha_{ij}, \qquad B := A^{-1}D^{-1} = (\beta^1, \dots, \beta^n),$$

so that

$$(e^{(n)})^T B = (e^{(n)})^T.$$

Consider now Equation (1) for $f \geqslant 0, \neq 0$, i.e., $y = Bg$, $g = Df \geqslant 0, \neq 0$. Without loss of generality we may assume that the sum of the components of g is 1. Then

$$(e^{(n)})^T y = (e^{(n)})^T Bg = (e^{(n)})^T g = 1,$$

for, by construction, $(e^{(n)})^T \beta^i = 1$, $\beta^i \geqslant 0$, $i = 1, \ldots, n$. That is, B is a nonsingular mapping of the simplex $\bar{S} := \{x \in R^n_+, (e^{(n)})^T x = 1\}$ into itself.

Now, the maximum principle states that the subsimplices composing the boundary of $\bar{S}$ are mapped into certain adjacent parts of $\bar{S}$: If $g = e^i$ then $y = \beta^i$, and, by the maximum principle,

$$y = \beta^i \in \bar{S}^{(i)} := \{x \in \bar{S},\ x_i \geqslant x_k,\ k = 1, \ldots, n\}.$$

Similarly, if g is a convex combination of e^i and e^j, $g \in \overline{\mathrm{co}}(e^i, e^j)$, then $y \in \overline{\mathrm{co}}(\beta^i, \beta^j) \in \bar{S}^{(i)} \cup \bar{S}^{(j)}$.

In general then, a monotone matrix A satisfies the maximum principle iff the corresponding matrix B has the property that

$$g \in \underset{i \in N_1}{\overline{\mathrm{co}}}\ (e^i) \quad \text{implies} \quad y = Bg \in \bigcup_{i \in N_1} \bar{S}^{(i)}$$

for all nonempty subsets N_1 of N.

These relations represent a geometrical interpretation of the maximum principle: The columns β^i of the matrix B (which is uniquely defined by A) are in $\bar{S}^{(i)}$ and form the corners of a nondegenerate simplex (in $\bar{S}$), the faces of which, spanned by some of the $\beta^{(i)}$, are in the union of the corresponding $\bar{S}^{(i)}$. This is, in fact, a geometrical characterization of (up to a positive diagonal scaling) the inverses to all monotone matrices satisfying the maximum principle. For inverses of M-matrices see, e.g., [7], [22].

Defining

$$\bar{S}^i(\varepsilon) := \{x \in \bar{S},\ x_i \geqslant \varepsilon \geqslant x_k,\ k = 1, \ldots, i-1, i+1, \ldots, n\},$$

we have $\bar{S}^{(i)}(\varepsilon) \subset \bar{S}^{(i)}$ for $0 \leqslant \varepsilon \leqslant 1/n$. Then, it is not difficult to see that $\beta^i \in \bar{S}^{(i)}(1/n)$, $i = 1, \ldots, n$, are sufficient conditions for the maximum principle to hold.

Obviously, for the equation $y = Bg$ singular matrices B may be admitted, too. For instance, the matrix with all entries equal to $1/n$ meets the above conditions.

5. APPLICATION TO SINGULAR MATRICES

For Equation (1) and the maximum principle to make sense in the case of a singular matrix A, the right-hand sides must be restricted to $R(A) \cap R^n_+$, with $R(A)$ the range of A, at least. Thus, we obtain a maximum principle depending on the matrix just considered.

Restricting f in (1) to $R(A^k) \cap R^n_+$, where $k = \operatorname{ind} A$, the index of A (see, e.g., [11]), we have the following result, in which $R(A^k) \cap R^n_+$ is tacitly assumed to be of dimension greater than 1, since otherwise the maximum principle would reduce to a trivial or empty statement.

THEOREM 2. *Let $A + \varepsilon I$ be nonsingular for $\varepsilon \in (0, \varepsilon_0]$ with some $\varepsilon_0 > 0$, and such that $(A + \varepsilon I)^{-1} f \geqslant 0$ for all $f \in R(A^k) \cap R^n_+$, $k = \operatorname{ind} A$. Then, if $A^{(-)} e^{(n)} \geqslant 0$, A satisfies the maximum principle for f restricted to $R(A^k) \cap R^n_+$.*

Proof. We proceed as in [20], the only difference being that now $\lim_{\varepsilon \to 0} (A + \varepsilon I)^{-1} f$ is assured to exist by $f \in R(A^k)$ (see [11]). ∎

That limit equals $A^D f$, with A^D the Drazin inverse of A, see [16]. We remark also that, in general, A^D will contain some negative entries: see [8], [15]. According to Theorem 2, such negative entries correspond to coordinate directions outside of $R(A^k) \cap R^n_+$.

6. RELATION TO THE MAXIMUM PRINCIPLE [3]

For discrete approximations to first kind elliptic or parabolic boundary value problems, of special interest is the case that A and f in (1) are of the form

$$A = \begin{pmatrix} I & 0 \\ B_1 & A_2 \end{pmatrix}, \qquad f = \begin{pmatrix} f^1 \\ 0 \end{pmatrix}, \tag{6}$$

where f^1 is an m-vector, I is the $m \times m$ identity, and A_2 is $(n - m) \times$

$(n-m)$, $1 \leqslant m < n$. Then, for $f^1 \geqslant 0$, the maximum principle [3] states that the solution y of (1), (6) satisfies

$$y \geqslant 0, \qquad \max_{m+1 \leqslant i \leqslant n} y_i \leqslant \max_{1 \leqslant j \leqslant m} y_j. \tag{7}$$

This property holds iff A^{-1} exists, $A^{-1} \geqslant 0$, and $-A_2^{-1} B_1 e^{(m)} \leqslant e^{(n-m)}$. Sufficient conditions are $A^{-1} \geqslant 0$ and $Ae^{(n)} \geqslant 0$: see [3].

For a general (not partitioned) monotone matrix A satisfying the maximum principle (1)–(2), the condition $Ae^{(n)} \geqslant 0$ is not sufficient (compare Example d in Section 8 below), but it is necessary, as will be shown now. For this aim, we use the notation of Section 4 and first formulate a lemma which can be proved by induction on n.

LEMMA. *Let $\alpha^1, \ldots, \alpha^n \in \bar{S}$, with components α^i_j fulfilling the relations*

$$\alpha^i_i \leqslant \frac{1}{n} \leqslant \alpha^i_1 = \cdots = \alpha^i_{i-1} = \alpha^i_{i+1} = \cdots = \alpha^i_n, \qquad i = 1, \ldots, n.$$

Then the midpoint $\bar{e}^{(n)} := (1/n) e^{(n)}$ of $\bar{S}$ is contained in $\overline{\mathrm{co}}_{i \in N}(\alpha^i)$.

THEOREM 3. *If A satisfies the maximum principle* (1)–(2), *then $A^{-1} \geqslant 0$ and $Ae^{(n)} \geqslant 0$.*

Proof. It is clear from the definition (see Section 2) that $A^{-1} \geqslant 0$. Proceeding from A to $B = (\beta^1, \ldots, \beta^n)$ as in Section 4, the inequality $Ae^{(n)} \geqslant 0$ can be rewritten equivalently as $\bar{e}^{(n)} \in \overline{\mathrm{co}}_{i \in N}(\beta^i)$.

Suppose first $n = 2$. By construction of B and from the maximum principle it follows that $1 \geqslant \beta^1_2 \geqslant \beta^1_2 \geqslant 0$ and $1 = \beta^1_1 + \beta^1_2$. Thus $\beta^1_1 \geqslant \frac{1}{2} \geqslant \beta^1_2$. Analogously, $\beta^2_2 \geqslant \frac{1}{2} \geqslant \beta^2_1$. Taking $\alpha^1 = \beta^2$ and $\alpha^2 = \beta^1$, the Lemma yields

$$\bar{e}^{(2)} \in \overline{\mathrm{co}}\,(\alpha^1, \alpha^2) = \overline{\mathrm{co}}\,(\beta^1, \beta^2).$$

Next, assume the theorem to be true for all dimensions l, $2 \leqslant l \leqslant n-1$. By the maximum principle, if $k \notin N^+(f)$ in the n-dimensional equation $y = Bg$ for some index k, then

$$\max_{\substack{i \in N \\ i \neq k}} y_i = \max_{\substack{j \in N^+(f) \\ j \neq k}} y_j.$$

This means that deleting the kth row and column in B will not destroy the properties of the maximum principle for the indices $i \neq k$. Multiplying the remaining $(n-1)\times(n-1)$ matrix from the right by

$$\tilde{D} := \operatorname{diag}(\tilde{d}_1, \dots, \tilde{d}_{k-1}, \tilde{d}_{k+1}, \dots, \tilde{d}_n), \qquad \tilde{d}_i := \left(1 - \beta_k^i\right)^{-1},$$

we arrive at a matrix $\tilde{B} = (\tilde{\beta}^1, \dots, \tilde{\beta}^{k-1}, \tilde{\beta}^{k+1}, \dots, \tilde{\beta}^n)$ with unit column sums. [Observe that $(1-\beta_k^i)^{-1}$ is well defined for $i \neq k$, since $(e^{(n)})^T \beta^i = 1$, $\beta^i \geqslant 0$, and $\beta_i^i \geqslant \beta_j^i$, $j = 1, \dots, n$, imply $\beta_k^i \leqslant \frac{1}{2}$, $i \neq k$.] Geometrically, this construction of $\tilde{B}$ from B consists in projecting in $\bar{S}$ all columns of B, except β^k, on $\bar{S} \cap \{x_k = 0\}$.

By induction, $\bar{e}^{(n-1)} \in \overline{\mathrm{co}}_{i \neq k}(\tilde{\beta}^i)$; say $\bar{e}^{(n-1)} = \sum_{i \neq k} s_i \tilde{\beta}^i$, with $\sum_{i \neq k} s_i = 1$ and $s_i \geqslant 0$. Now, put

$$\gamma_k := \sum_{i \neq k} s_i \tilde{d}_i \quad \text{and} \quad \delta_i^{(k)} := s_i \tilde{d}_i \gamma_k^{-1}, \quad i \neq k.$$

Then $\delta_i^{(k)} \geqslant 0$ and $\sum_{i \neq k} \delta_i^{(k)} = 1$. Hence $\bar{S} \ni \sum_{i \neq k} \delta_i^{(k)} \beta^i =: \alpha^k$ and $\alpha_1^k = \cdots = \alpha_{k-1}^k = \alpha_{k+1}^k = \cdots = \alpha_n^k$.

Let $g^k := \sum_{i \neq k} \delta_i^{(k)} e^i$. Then $\alpha^k = B g^k$, and from the maximum principle, since e^k is missing in g^k, it follows that $\alpha_k^k \leqslant \alpha_i^k$, $i \neq k$. Then $\alpha_k^k \leqslant 1/n \leqslant \alpha_i^k$, $i \neq k$, due to $(e^{(n)})^T \alpha^k = 1$. In this way, a set of vectors $\{\alpha^1, \dots, \alpha^n\}$ is obtained to which the Lemma applies. Thus $\bar{e}^{(n)} \in \overline{\mathrm{co}}_{i \in N}(\alpha^i) \subseteq \overline{\mathrm{co}}_{i \in N}(\beta^i)$, completing the induction. ■

Combining Theorems 1 and 3, we get a corollary which is of interest for both economic and boundary-value problems.

Corollary. *An M-matrix A satisfies the maximum principle* (1)–(2) *iff* $Ae^{(n)} \geqslant 0$.

7. A GENERALIZED MAXIMUM PRINCIPLE

For (1), (6) we may define a more general form of the maximum principle (6), (7): Let c_0 be a positive number which may depend on A. The matrix (6) is said to satisfy the maximum principle with constant c_0 if (1), (6) and

$f^1 \geqslant 0$ imply

$$y \geqslant 0, \qquad \max_{m+1 \leqslant i \leqslant n} y_i \leqslant c_0 \max_{1 \leqslant j \leqslant m} y_j. \tag{8}$$

For this generalized maximum principle to hold, in analogy to [3], the conditions $A^{-1} \geqslant 0$ and $-A_2^{-1}B_1e^{(m)} \leqslant c_0e^{(n-m)}$ are now shown to be necessary and sufficient.

In correspondence to the partitioning (6) we write $y = (y^1, y^2)^T$ for the solution of (1), (6). Obviously, $A^{-1} \geqslant 0$ is necessary; hence $A_2^{-1} \geqslant 0$, $-A_2^{-1}B_1 \geqslant 0$. By the maximum principle,

$$c_0e^{(n-m)} \max_{1 \leqslant i \leqslant m} y_i^1 \geqslant y^2 = -A_2^{-1}B_1y^1.$$

In particular, for $f^1 = e^{(m)} = y^1$, $c_0e^{(n-m)} \geqslant -A_2^{-1}B_1e^{(m)}$.

On the other hand, let $A^{-1} \geqslant 0$ and $-A_2^{-1}B_1e^{(m)} \leqslant c_0e^{(n-m)}$. Then y is uniquely determined, is nonnegative for $f^1 = y^1 \geqslant 0$, and fulfils (8):

$$y^2 = -A_2^{-1}B_1y^1 \leqslant \max_{1 \leqslant i \leqslant m} y_i^1\left(-A_2^{-1}B_1e^{(m)}\right) \leqslant c_0 \max_{1 \leqslant i \leqslant m} y_i^1e^{(n-m)}.$$

From this it follows also that for a given monotone matrix (6) the best possible c_0 is

$$c_0 = \max_{m+1 \leqslant i \leqslant n} \left(-A_2^{-1}B_1e^{(m)}\right)_i. \tag{9}$$

Usually it will be simpler to determine a constant c_0 such that $B_1e^{(m)} + c_0A_2e^{(n-m)} \geqslant 0$, which is equivalent to

$$Ac \geqslant 0 \qquad \text{for} \quad c = \left(e^{(m)}, c_0e^{(n-m)}\right)^T. \tag{10}$$

Remark. If (1), (6) correspond to some monotone approximation

$$L_hy = 0, \quad x \in \Omega_h, \qquad \text{with } y \text{ specified on } \partial\Omega_h,$$

of a monotone boundary-value problem

$$Lu = 0, \quad x \in \Omega, \qquad u = g, \quad x \in \partial\Omega,$$

then (8), (9) mean just the estimate

$$\|y\|_{C(\Omega_h)} \leqslant \|Y\|_{C(\Omega_h)} \|y\|_{C(\partial\Omega_h)}, \tag{11}$$

where $\|\cdot\|_{C(\omega)}$ is the maximum norm over a finite set ω, and Y is the solution of

$$L_h Y = 0, \quad x \in \Omega_h, \qquad Y = 1, \quad x \in \partial\Omega_h.$$

8. EXAMPLES

We give now some examples.

EXAMPLE a. The matrix

$$A_\alpha := \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & \alpha \\ -1 & -1 & 4 \end{pmatrix}$$

is monotone for $\frac{1}{4} \geqslant \alpha > -11$, with $A^{(-)}$ nonsingular for $\alpha > -11$, and $0 \leqslant A^{(-)} e^{(3)}$ for $\alpha \geqslant -3$. According to Theorem 1, the matrix A satisfies the maximum principle for $\alpha \in [-3, \frac{1}{4}]$. Using the necessary and sufficient conditions of Section 4, it can be verified that the maximum principle does not hold for other values of α.

If $\frac{1}{4} > \alpha > -11$, then there is an $\varepsilon_0(\alpha) > 0$ such that $(A_\alpha + \varepsilon I)^{-1} > 0$ for $\varepsilon \in [0, \varepsilon_0]$. If $\alpha = \frac{1}{4}$ then $(A_\alpha + \varepsilon I)^{-1} \ngeqslant 0$ for $\varepsilon > 0$.

EXAMPLE b.

$$A_\beta := \begin{pmatrix} 1 & -1 & \beta \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

is monotone for $0 < \beta \leqslant 1$ but not quasidominant (compare Remark 2 to Theorem 1) there: $M^{-1}(A_\beta) \leqslant 0$. Theorem 1 assures the maximum principle will hold for $\beta \in (0, 1]$, since $A_\beta^{(-)}$ is singular and irreducible, and $A_\beta^{(-)} e^{(3)} \geqslant 0$, for $\beta \geqslant 0$. Further, if $0 < \beta < 1$, then there exists $\varepsilon_0(\beta) > 0$ with $(A_\beta + \varepsilon I)^{-1} \geqslant 0$ for $\varepsilon \in [0, \varepsilon_0]$, but $(A_\beta + \varepsilon I)^{-1} \ngeqslant 0$ for $\beta = 1$ and $\varepsilon > 0$.

EXAMPLE c. Let

$$A := \begin{pmatrix} 4 & 2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix}, \qquad A^D = \frac{1}{6}\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 \end{pmatrix}.$$

Here $A = A^{(-)}$, $A^{(-)}e^{(4)} \geqslant 0$, $(A+\varepsilon I)^{-1} \geqslant 0$ for $\varepsilon > 0$ (since then $A + \varepsilon I$ is an M-matrix), $\operatorname{ind} A = 1$, and $R(A) \cap R_+^4 = \{x_3 = 0\} \cap R_+^4$. By Theorem 2, the maximum principle will hold for A if the right-hand sides are restricted to the face $x_3 = 0$ of R_+^4.

EXAMPLE d. Let E be the $n \times n$ matrix with all entries equal to 1, $n > 1$, and $P = (p_{ij})$ a symmetric permutation matrix which is not the identity. For $0 < \delta \leqslant 1$, consider

$$A_\delta := \delta^{-1}[P - (1-\delta)\delta_n E], \qquad \delta_n := [(1-\delta)n + \delta]^{-1}.$$

This matrix is monotone: $A_\delta^{-1} = \delta P + (1-\delta)E$.

We have

$$A_\delta e^{(n)} = \delta_n > 0, \quad \text{but} \quad (A^{(-)}e^{(n)})_i = \begin{cases} \delta_n, & p_{ii} = 1, \\ -(1-\delta)(n-1)\delta_n, & p_{ii} = 0. \end{cases}$$

Therefore, if $0 < \delta < 1$, then there are at least two rows with negative sums in $A^{(-)}$. If $\delta = 1$, $A^{(-)}$ becomes singular and reducible. Thus, Theorem 1 is not applicable.

In fact, the maximum principle does not hold: Let $p_{ij} = 1$ for some $i \neq j$. Then the solution to $A_\delta y = f = e^j$ is $y = \delta e^i + (1-\delta)e^{(n)}$, i.e.,

$$y_i = 1 = \max_{k \in N} y_k, \quad \text{but} \quad y_j = 1 - \delta = \max_{l \in N^+(f)} y_l.$$

In other words, if A is any matrix satisfying $A^{-1} \geqslant 0$ and $Ae^{(n)} \geqslant 0$, then in general, there is no constant c_0 such that $\max_{k \in N^0(f)} y_k \leqslant c_0 \max_{l \in N^+(f)} y_l$ for the solution y of (1).

EXAMPLE e. We consider the boundary-value problem

$$-u'' + d(x)u = 0, \quad 0 < x < 1, \qquad u(0) = u_0, \quad u(1) = u_1,$$

where d is assumed to satisfy $d(x) \geqslant -D_0 > -8$, $D_0 > 0$, and its approximation

$$-y_{\bar{x}x,i} + d_i y_i = 0, \quad i = 2,\dots,n-1, \qquad y_1 = u_0, \quad y_n = u_1. \tag{12}$$

Here $y_i := y(x_i)$, $x_i := (i-1)h$, $h := 1/(n-1)$, $n \geqslant 3$, and $y_{\bar{x}x,i} := (y_{i+1} - 2y_i + y_{i-1})/h^2$. The linear system (1) corresponding to (12) has, after appropriate rearrangement, a matrix $A = A(d)$ of the form (6).

Let $d^- = \min(0, d(x))$. The matrix $A_2(0)$ fulfils

$$\|A_2^{-1}(0)\|_\infty \leqslant \tfrac{1}{8},$$

and therefore

$$\|A_2^{-1}(d^-)\|_\infty \leqslant \frac{1}{8 - D_0}; \tag{13}$$

see, e.g., [2].

The matrices of the family $\{A(\lambda d^-),\ 0 \leqslant \lambda \leqslant 1\}$ have the following properties: Their off-diagonal entries are nonpositive; their diagonal entries are positive; all matrices are nonsingular, as follows from (13); and $A(0)$ is an M-matrix. Hence, by a result of [14], all matrices of the family are M-matrices; in particular, $A(d^-)$ is. Since $A(d) \geqslant A(d^-)$, it follows then that $A(d)$ is an M-matrix, too, and that $0 \leqslant A^{-1}(d) \leqslant A^{-1}(d^-)$.

Now, the Corollary yields that the maximum principle (1)–(2) is satisfied iff $d(x) \geqslant 0$. In case $d(x) \geqslant d_0 \geqslant 0$ and $n > 3$, via (9) the generalized principle (8) is found to hold with constant $c_0 = 1/(1 + d_0 h^2)$.

For the maximum principle [3] to be true, $d(x)$ may have small negative values if these are balanced by sufficiently great positive values.

To obtain an estimate of the function $Y = Y(d)$ figuring in the inequality (11), where now $\partial\Omega_h = \{0,1\}$ and $\Omega_h = \{x_i, i = 2,\dots,n-1\}$, we use $0 \leqslant Y(d) \leqslant Y(d^-)$ and put $Y(d^-) = 1 + z$, $z = z(d^-)$:

$$-z_{\bar{x}x,i} + d_i^- z_i = -d_i^-, \quad i = 2,\dots,n-1, \qquad z_1 = z_n = 0.$$

Then, by (13),

$$\|z\|_{C(\Omega_h)} \leqslant \frac{D_0}{8 - D_0}, \qquad \text{i.e.,} \quad \|Y(d)\|_{C(\Omega_h)} \leqslant \frac{8}{8 - D_0}.$$

Thus, if $d(x) \geqslant -D_0 > -8$, the generalized maximum principle (8), (9) for the problem (12) gives the estimate

$$\|y\|_{C(\Omega_h)} \leqslant \frac{8}{8 - D_0} \max(|u_0|, |u_1|).$$

9. EXTENSION TO H-MATRICES

Finally, we consider a maximum principle for matrices which are not necessarily monotone.

The matrix A is said to satisfy the maximum principle for the absolute values if

$$N^+(|f|) = \varnothing \quad \Rightarrow \quad y = 0,$$

and

$$N^+(|f|) \neq \varnothing \quad \Rightarrow \quad \max_{i \in N^0(|f|)} |y_i| < \max_{j \in N^+(|f|)} |y_j|$$

for all y and f connected by (1). Here $|f|$ is the vector of the absolute values $|f_k|$, $k = 1, \ldots, n$.

THEOREM 4. *The necessary and sufficient condition for* $A = (a_{ij})$ *to satisfy the maximum principle for the absolute values is*

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \qquad i = 1, \ldots, n. \tag{14}$$

In other words, A is to be a strictly diagonally dominant H-matrix.

Proof. Under the above condition (14), it is well known that A is nonsingular (see e.g. [6]), i.e., $N^+(|f|) = \varnothing$ implies $y = 0$. Using the same reasoning, it follows in case $N^+(|f|) \neq \varnothing$ that $k \notin N^0(|f|)$ for the maximal component $|y_k|$ of $|y|$.

On the other hand, if A satisfies the maximum principle for the absolute values, then $y = e^i$ for some i shows that $a_{ii} \neq 0$. Next, take

$$y_j = \operatorname{sign}(a_{ij}), \quad j \neq i, \qquad y_i = \left(1 - \frac{\sum_{j \neq i} |a_{ij}| - |a_{ii}|}{|a_{ii}|}\right) \operatorname{sign}(a_{ii}).$$

Then $(Ay)_i = 0$, i.e. $i \in N^0(|Ay|)$, and now

$$|y_i| \leqslant \max_{l \in N^0} |y_l| < \max_{j \in N^+} |y_j| = 1$$

shows (14) to hold. ■

This proof is essentially that of [21], where properties of the solution of (1) with tridiagonal matrix and $N^0 = \{2, \ldots, n-1\}$ have been investigated.

By continuity, nonsingular matrices fulfilling (14) with equality admitted will satisfy the corresponding weaker maximum principle.

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